

SATBILITY OF TERNARY HOMOMORPHISMS VIA GENERALIZED JENSEN EQUATION

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ABSTRACT. In this paper, we establish the generalized Hyers–Ulam–Rassias stability of homomorphisms between ternary algebras associated to the generalized Jensen functional equation $rf(\frac{sx+ty}{r}) = sf(x) + tf(y)$.

1. INTRODUCTION

A ternary (associative) algebra $(\mathcal{A}, [\])$ is a linear space \mathcal{A} over a scalar field $\mathbb{F} = \mathbb{R}$ or \mathbb{C} equipped with a linear mapping, the so-called ternary product, $[\] : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ such that $[[abc]de] = [a[bcd]e] = [ab[cde]]$ for all $a, b, c, d, e \in \mathcal{A}$. This notion is a natural generalization of the binary case. Indeed if (\mathcal{A}, \odot) is a usual (binary) algebra then $[abc] := (a \odot b) \odot c$ induced a ternary product making \mathcal{A} into a ternary algebra which will be called trivial. It is known that unital ternary algebras are trivial and finitely generated ternary algebras are ternary subalgebras of trivial ternary algebras [3]. There are other types of ternary algebras in which one may consider other versions of associativity (see [15]). Some examples of ternary algebras are (i) “cubic matrices” introduced by Cayley [5] which was in turn generalized by Kapranov, Gelfand and Zelevinskii [11]; (ii) the ternary algebra of the polynomials of odd degrees in one variable equipped with the ternary operation $[p_1 p_2 p_3] = p_1 \odot p_2 \odot p_3$, where \odot denotes the usual multiplication of polynomials. By a Banach ternary algebra we mean a ternary algebra equipped with a complete norm $\| \cdot \|$ such that $\|[abc]\| \leq \|a\| \|b\| \|c\|$.

The stability problem of functional equations originated from a question of S. Ulam [23], posed in 1940, concerning the stability of group homomorphisms. In 1941, D. H. Hyers [8] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th. M. Rassias [20] extended the theorem of Hyers by considering the unbounded Cauchy difference $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$, ($\varepsilon > 0, p \in [0, 1)$). The result of

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Rassias, which is also true for $p < 0$, has provided a lot of influence in the development of what we now call *Hyers–Ulam–Rassias stability* of functional equations. In 1992, a generalization of Th. M. Rassias’ theorem, the so-called generalized Hyers–Ulam–Rassias stability, was obtained by Găvruta [7] by following the same approach as in [20]. During the last decades several stability problems of functional equations have been investigated in the spirit of Hyers–Ulam–Rassias–Găvruta. See [6, 9, 21, 16] and references therein for more detailed information on stability of functional equations.

Stability of algebraic and topological homomorphisms has been investigated by many mathematicians, for an extensive account on the subject see [22]. In [2] the authors investigated the stability of homomorphisms between J^* -algebras associated to the Cauchy equation. Some results on stability ternary homomorphisms may be found at [1, 15]. C. Park [17, 18] studied the stability of Poisson C^* -homomorphisms and JB^* -homomorphisms associated to the Jensen equation $2f(\frac{x+y}{2}) = f(x) + f(y)$ where f is a mapping between linear spaces. The generalized stability of this equation was studied by K. Jun and Y. Lee [12] and also [13].

A generalization of the Jensen equation is the equation

$$rf(\frac{sx+ty}{r}) = sf(x) + tf(y).$$

where f is a mapping between linear spaces and r, s, t are given constant values (see also [14]). It is easy to see that a mapping $f : X \rightarrow Y$ between linear spaces with $f(0) = 0$ satisfies the generalized Jensen equation if and only if it is additive; cf. [4].

In this paper, using some ideas from [2, 17, 15], we establish the generalized Hyers–Ulam–Rassias stability of ternary homomorphisms associated to the generalized Jensen functional equation. If a ternary algebra $(\mathcal{A}, [\])$ has an identity, i.e. an element e such that $a = [aee] = [eaa] = [eea]$ for all $a \in \mathcal{A}$, then $a \odot b := [aeb]$ is a binary product for which we have

$$(a \odot b) \odot c = [[aeb]ec] = [ae[bec]] = a \odot (b \odot c)$$

and

$$a \odot e = [aee] = a = [eea] = e \odot a,$$

for all $a, b, c \in \mathcal{A}$ and so $(\mathcal{A}, [\])$ may be considered as a (binary) algebra. Conversely, if (\mathcal{A}, \odot) is any (binary) algebra, then $[abc] := a \odot b \odot c$ makes \mathcal{A} into a ternary algebra with

the unit e such that $a \odot b = [aeb]$. Thus our results may be applied to investigate of stability of algebra homomorphisms; see [19].

Throughout the paper, \mathcal{A} denotes a ternary algebra, \mathcal{B} is a Banach ternary algebra, X denotes a linear space and Y represents a Banach space. In addition, we assume r, s, t to be constant positive integers. If a mapping f satisfies the generalized Jensen equation, then so does $f(x) - f(0)$. Hence without lose of generality we can assume that $f(0) = 0$.

2. MAIN RESULTS

In this section, we are going to establish the generalized Hyers–Ulam–Rassias stability of homomorphisms between ternary algebras associated with the generalized Jensen functional equation. We start with study of stability of generalized Jensen equation using different Hyers' sequences from those of [13] and [4].

Theorem 2.1. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X \times X \rightarrow [0, \infty)$ satisfying*

$$\tilde{\varphi}(x, y) := \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r}{s}\right)^{-n} \varphi\left(\left(\frac{r}{s}\right)^n x, \left(\frac{r}{s}\right)^n y\right) < \infty,$$

and

$$(2.1) \quad \left\| r f\left(\frac{sx + ty}{r}\right) - sf(x) - tf(y) \right\| \leq \varphi(x, y),$$

for all $x, y \in X$. Then there exists a unique additive mapping $T : X \rightarrow Y$ given by $T(x) := \lim_{n \rightarrow \infty} \left(\frac{r}{s}\right)^{-n} f\left(\left(\frac{r}{s}\right)^n x\right)$ such that

$$\|f(x) - T(x)\| \leq \tilde{\varphi}(x, x)$$

for all $x \in X$.

Proof. Set $y = 0$ in 2.1 to get

$$\left\| r f\left(\frac{sx}{r}\right) - sf(x) \right\| \leq \varphi(x, 0),$$

whence

$$\left\| f(x) - \left(\frac{r}{s}\right)^{-1} f\left(\frac{r}{s}x\right) \right\| \leq \frac{1}{r} \varphi\left(\frac{r}{s}x, 0\right),$$

for all $x \in X$. Assume that for some positive integer n

$$\|f(x) - (\frac{r}{s})^{-n} f((\frac{r}{s})^n x)\| \leq \frac{1}{r} \sum_{k=0}^{n-1} (\frac{r}{s})^{-k} \varphi((\frac{r}{s})^{k+1} x, 0),$$

for all $x \in X$. Then

$$\begin{aligned} \|f(x) - (\frac{r}{s})^{-n-1} f((\frac{r}{s})^{n+1} x)\| &\leq \|f(x) - (\frac{r}{s})^{-n} f((\frac{r}{s})^n x)\| \\ &\quad + \|(\frac{r}{s})^{-n} f((\frac{r}{s})^n x) - (\frac{r}{s})^{-n} (\frac{r}{s})^{-1} f((\frac{r}{s})(\frac{r}{s})^n x)\| \\ &\leq \|f(x) - (\frac{r}{s})^{-n} f((\frac{r}{s})^n x)\| \\ &\quad + (\frac{r}{s})^{-n} \|f((\frac{r}{s})^n x) - (\frac{r}{s})^{-1} f((\frac{r}{s})(\frac{r}{s})^n x)\| \\ &\leq \frac{1}{r} \sum_{k=0}^{n-1} (\frac{r}{s})^{-k} \varphi((\frac{r}{s})^{k+1} x, 0) \\ &\quad + (\frac{r}{s})^{-n} \frac{1}{r} \varphi((\frac{r}{s})^{n+1} x, 0) \\ &\leq \frac{1}{r} \sum_{k=0}^n ((\frac{r}{s})^{-k} \varphi((\frac{r}{s})^{k+1} x, 0)) \end{aligned}$$

for all $x \in X$. Using the induction, we conclude that

$$(2.2) \quad \|f(x) - (\frac{r}{s})^{-n} f((\frac{r}{s})^n x)\| \leq \frac{1}{r} \sum_{k=0}^{n-1} (\frac{r}{s})^{-k} \varphi((\frac{r}{s})^{k+1} x, 0),$$

for all $x \in X$ and all $n \in \mathbb{N}$. Similarly one can show that

$$\|(\frac{r}{s})^{-n} f((\frac{r}{s})^n x) - (\frac{r}{s})^{-m} f((\frac{r}{s})^m x)\| \leq \frac{1}{r} \sum_{k=m}^{n-1} (\frac{r}{s})^{-k} \varphi((\frac{r}{s})^{k+1} x, 0),$$

for all positive integers $n > m$ and all $x \in X$. Hence the sequence $\{(\frac{r}{s})^{-n} f((\frac{r}{s})^n x)\}$ is Cauchy and so is convergent in the complete space Y . Thus we can define the mapping $T : X \rightarrow Y$ by

$$(2.3) \quad T(x) := \lim_{n \rightarrow \infty} (\frac{r}{s})^{-n} f((\frac{r}{s})^n x)$$

Replace x, y by $(\frac{r}{s})^n x, (\frac{r}{s})^n y$, respectively, in 2.1 to obtain

$$\begin{aligned} \|r(\frac{r}{s})^{-n} f(\frac{s(\frac{r}{s})^n x - t(\frac{r}{s})^n y}{r}) - s(\frac{r}{s})^{-n} f((\frac{r}{s})^n x) + t(\frac{r}{s})^{-n} f((\frac{r}{s})^n y)\| \\ \leq (\frac{r}{s})^{-n} \varphi((\frac{r}{s})^n x, (\frac{r}{s})^n y), \end{aligned}$$

for all $x \in X$ and all n . Letting $n \rightarrow \infty$, we deduce that T satisfies the generalized Jensen functional equation and so it is additive. In addition 2.2 and 2.3 yield

$$\|f(x) - T(x)\| \leq \tilde{\varphi}(x, x)$$

for all $x \in X$.

We use a standard technique to prove the uniqueness assertion (see e.g. [16]). First note that for all positive integer j we have

$$\begin{aligned} \left(\frac{r}{s}\right)^j T(x) &= \left(\frac{r}{s}\right)^j \lim_{n \rightarrow \infty} \left(\frac{r}{s}\right)^{-n} f\left(\left(\frac{r}{s}\right)^n x\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{r}{s}\right)^{j-n} f\left(\left(\frac{r}{s}\right)^{n-j} \left(\left(\frac{r}{s}\right)^j x\right)\right) \\ &= T\left(\left(\frac{r}{s}\right)^j x\right) \end{aligned}$$

Now let T' be another additive mapping satisfying $\|f(x) - T'(x)\| \leq \tilde{\varphi}(x, x)$ for all $x \in X$. Then

$$\begin{aligned} \|T(x) - T'(x)\| &= \left(\frac{r}{s}\right)^{-j} \|T\left(\left(\frac{r}{s}\right)^j x\right) - T'\left(\left(\frac{r}{s}\right)^j x\right)\| \\ &\leq \left(\frac{r}{s}\right)^{-j} \|T\left(\left(\frac{r}{s}\right)^j x\right) - f\left(\left(\frac{r}{s}\right)^j x\right)\| + \left(\frac{r}{s}\right)^{-j} \|f\left(\left(\frac{r}{s}\right)^j x\right) - T'\left(\left(\frac{r}{s}\right)^j x\right)\| \\ &\leq 2\left(\frac{r}{s}\right)^{-j} \tilde{\varphi}\left(\left(\frac{r}{s}\right)^j x, \left(\frac{r}{s}\right)^j x\right) \\ &= 2\left(\frac{r}{s}\right)^{-j} \sum_{k=0}^{\infty} \left(\frac{r}{s}\right)^{-k} \varphi\left(\left(\frac{r}{s}\right)^k \left(\frac{r}{s}\right)^j x, \left(\frac{r}{s}\right)^k \left(\frac{r}{s}\right)^j x\right) \\ &= 2 \sum_{k=j}^{\infty} \left(\frac{r}{s}\right)^{-k} \varphi\left(\left(\frac{r}{s}\right)^k x, \left(\frac{r}{s}\right)^k x\right) \end{aligned}$$

for all $x \in X$. The right hand side tends to zero as $j \rightarrow \infty$, hence $T(x) = T'(x)$ for all $x \in X$. \square

In a similar fashion one may prove the following theorem.

Theorem 2.2. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X \times X \rightarrow [0, \infty)$ satisfying*

$$\tilde{\varphi}(x, y) := \frac{1}{s} \sum_{n=0}^{\infty} \left(\frac{r}{s}\right)^n \varphi\left(\left(\frac{r}{s}\right)^{-n} x, \left(\frac{r}{s}\right)^{-n} y\right) < \infty,$$

and

$$\|rf(\frac{sx+ty}{r}) - sf(x) - tf(y)\| \leq \varphi(x, y),$$

for all $x, y \in X$. Then there exists a unique additive mapping $T : X \rightarrow Y$ given by $T(x) := \lim_{n \rightarrow \infty} (\frac{r}{s})^n f((\frac{r}{s})^{-n}x)$ such that

$$\|f(x) - T(x)\| \leq \tilde{\varphi}(x, x)$$

for all $x \in X$.

The following proposition gives a sufficient condition in order a mapping satisfying a Jensen type inequality really to be a ternary homomorphism.

Proposition 2.3. *Let $r \neq s$ and $T : \mathcal{A} \rightarrow \mathcal{B}$ be a mapping with $T(\frac{r}{s}x) = \frac{r}{s}T(x)$, $x \in \mathcal{A}$ for which there exists a function $\varphi : \mathcal{A}^5 \rightarrow [0, \infty)$ satisfying*

$$\lim_{n \rightarrow \infty} (\frac{r}{s})^{-n} \varphi((\frac{r}{s})^n x, (\frac{r}{s})^n y, (\frac{r}{s})^n u, (\frac{r}{s})^n v, (\frac{r}{s})^n w) = 0,$$

and

$$(2.4) \quad \begin{aligned} & \|rT(\frac{\mu sx + \mu ty + [uvw]}{r}) - \mu sT(x) + \mu tT(y) - [T(u)T(v)T(w)]\| \\ & \leq \varphi(x, y, u, v, w), \end{aligned}$$

for all $\mu \in \mathbb{C}$, $x, y, u, v, w \in \mathcal{A}$.

Proof. $T(0) = 0$ since $T(0) = \frac{r}{s}T(0)$ and $\frac{r}{s} \neq 1$. Putting $\mu = 1$, $u = v = w = 0$ and replacing x, y by $(\frac{r}{s})^n x, (\frac{r}{s})^n y$ in 2.4, we get

$$\begin{aligned} & \|r(\frac{r}{s})^{-n}T((\frac{r}{s})^n \frac{sx+ty}{r}) - s(\frac{r}{s})^{-n}T((\frac{r}{s})^n x) + t(\frac{r}{s})^{-n}T((\frac{r}{s})^n y)\| \\ & \leq (\frac{r}{s})^{-n} \varphi((\frac{r}{s})^n x, (\frac{r}{s})^n y, 0, 0, 0), \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ we conclude that T satisfies the Jensen equation. Hence T is additive. Similarly one can prove that $T(\mu x) = \mu T(x)$ for all $\mu \in \mathbb{C}$, $x \in \mathcal{A}$.

Set $x = y = 0$ and replace u, v, w by $(\frac{r}{s})^n u, (\frac{r}{s})^n v, (\frac{r}{s})^n w$ in 2.4. Then

$$\begin{aligned} \|rT(\frac{[uvw]}{r}) - [T(u)T(v)T(w)]\| &= (\frac{r}{s})^{-3n} \|rT(\frac{[(\frac{r}{s})^n u((\frac{r}{s})^n v)(\frac{r}{s})^n w]}{r}) \\ &\quad - [T((\frac{r}{s})^n u)T((\frac{r}{s})^n v)T((\frac{r}{s})^n w)]\| \\ &\leq (\frac{r}{s})^{-3n} \varphi(0, 0, (\frac{r}{s})^n u, (\frac{r}{s})^n v, (\frac{r}{s})^n w) \\ &\leq (\frac{r}{s})^{-n} \varphi(0, 0, (\frac{r}{s})^n u, (\frac{r}{s})^n v, (\frac{r}{s})^n w), \end{aligned}$$

for all $u, v, w \in \mathcal{A}$. The right hand side tends to zero as $n \rightarrow \infty$, so that $T([uvw]) = rT(\frac{[uvw]}{r}) = [T(u)T(v)T(w)]$ for all $u, v, w \in \mathcal{A}$. Thus T is a ternary homomorphism. \square

Theorem 2.4. *Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a mapping such that $f(0) = 0$, and there exists a function $\varphi : \mathcal{A}^5 \rightarrow [0, \infty)$ such that*

$$(2.5) \quad \tilde{\varphi}(x, y, u, v, w) := \frac{1}{r} \sum_{j=0}^{\infty} (\frac{r}{s})^{-j} \varphi((\frac{r}{s})^j x, (\frac{r}{s})^j y, (\frac{r}{s})^j u, (\frac{r}{s})^j v, (\frac{r}{s})^j w) < \infty,$$

and

$$(2.6) \quad \begin{aligned} \|rf(\frac{\mu sx + \mu ty + [uvw]}{r}) - \mu sf(x) + \mu tf(y) - [f(u)f(v)f(w)]\| \\ \leq \varphi(x, y, u, v, w), \end{aligned}$$

for all $\mu \in \mathbb{C}, x, y, u, v, w \in \mathcal{A}$. Then there exists a unique ternary homomorphism $T : \mathcal{A} \rightarrow \mathcal{B}$ such that

$$(2.7) \quad \|f(x) - T(x)\| \leq \tilde{\varphi}(x, x, 0, 0, 0)$$

for all $x \in \mathcal{A}$.

Proof. Put $u = v = w = 0$ and $\mu = 1 \in \mathbb{T}^1$ in 2.6. It follows from theorem 2.1 that there is a unique additive mapping $T : \mathcal{A} \rightarrow \mathcal{B}$ defined by

$$T(x) = \lim_{n \rightarrow \infty} (\frac{r}{s})^{-n} f((\frac{r}{s})^n x),$$

and satisfying 2.7 for all $x \in \mathcal{A}$.

Let $\mu \in \mathbb{T}^1$. Replacing x by $(\frac{r}{s})^{n+1}x$ and y by 0 in 2.6, we get

$$\|f((\frac{r}{s})^n \mu x) - \mu (\frac{r}{s})^{-1} f((\frac{r}{s})^{n+1} x)\| \leq \frac{1}{r} \varphi((\frac{r}{s})^{n+1} x, 0, 0, 0, 0),$$

for all $x \in \mathcal{A}$. It follows from $\|rf(x) - sf(\frac{r}{s}x)\| \leq \varphi(\frac{r}{s}x, 0, 0, 0, 0)$ that

$$\|f((\frac{r}{s})^n x) - (\frac{r}{s})^{-1} f((\frac{r}{s})^{n+1} x)\| \leq \frac{1}{r} \varphi((\frac{r}{s})^{n+1} x, 0, 0, 0, 0),$$

Hence

$$\begin{aligned} \|(\frac{r}{s})^{-n} f((\frac{r}{s})^n \mu x) - \mu(\frac{r}{s})^{-n} f((\frac{r}{s})^n x)\| &\leq (\frac{r}{s})^{-n} \|f((\frac{r}{s})^n \mu x) - \mu(\frac{r}{s})^{-1} f((\frac{r}{s})^{n+1} x)\| \\ &\quad + (\frac{r}{s})^{-n} \|\mu f((\frac{r}{s})^n x) - \mu(\frac{r}{s})^{-1} f((\frac{r}{s})^{n+1} x)\| \\ &\leq 2(\frac{r}{s})^{-n} (\frac{1}{r}) \varphi((\frac{r}{s})^{n+1} x, 0, 0, 0, 0), \end{aligned}$$

for all $x \in \mathcal{A}$. Taking the limit and using 2.3 and noting that the right hand side tends to zero as $n \rightarrow \infty$, we infer that

$$T(\mu x) = \lim_{n \rightarrow \infty} (\frac{r}{s})^{-n} f((\frac{r}{s})^n \mu x) = \lim_{n \rightarrow \infty} (\mu(\frac{r}{s})^{-n} f((\frac{r}{s})^n x)) = \mu T(x),$$

for all $x \in \mathcal{A}$. Obviously, $T(0x) = 0 = 0T(x)$.

Next, let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and let M be a natural number greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = 1/3$. By Theorem 1 of [10], there exist three numbers $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. By the additivity of T we get $T(\frac{1}{3}x) = \frac{1}{3}T(x)$ for all $x \in \mathcal{A}$. Therefore,

$$\begin{aligned} T(\lambda x) &= T(\frac{M}{3} \cdot 3 \cdot \frac{\lambda}{M} x) = MT(\frac{1}{3} \cdot 3 \cdot \frac{\lambda}{M} x) = \frac{M}{3} T(3 \cdot \frac{\lambda}{M} x) \\ &= \frac{M}{3} T(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3} (T(\mu_1 x) + T(\mu_2 x) + T(\mu_3 x)) \\ &= \frac{M}{3} (\mu_1 + \mu_2 + \mu_3) T(x) = \frac{M}{3} \cdot 3 \cdot \frac{\lambda}{M} \\ &= \lambda T(x), \end{aligned}$$

for all $x \in \mathcal{A}$. So that T is \mathbb{C} -linear.

Set $\mu = 1$ and $x = y = 0$, and replace u, v, w by $(\frac{r}{s})^n u, (\frac{r}{s})^n v, (\frac{r}{s})^n w$, respectively, in (2.6). Then

$$\begin{aligned} (\frac{r}{s})^{-3n} \|rf((\frac{r}{s})^{3n} \frac{[uvw]}{r}) - [f((\frac{r}{s})^n u) f((\frac{r}{s})^n v) f((\frac{r}{s})^n w)]\| \\ \leq (\frac{r}{s})^{-3n} \varphi(0, 0, (\frac{r}{s})^n u, (\frac{r}{s})^n v, (\frac{r}{s})^n w), \end{aligned}$$

for all $u, v, w \in \mathcal{A}$. Then by applying the continuity of the ternary product $(x, y, z) \mapsto [xyz]$ we deduce

$$\begin{aligned} T([uvw]) &= rT\left(\frac{1}{r}[uvw]\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{r}{s}\right)^{-3n} r f\left(\left(\frac{r}{s}\right)^{3n} \frac{[uvw]}{r}\right) \\ &= \lim_{n \rightarrow \infty} \left[\left(\frac{r}{s}\right)^{-n} f\left(\left(\frac{r}{s}\right)^n u\right) \left(\frac{r}{s}\right)^{-n} f\left(\left(\frac{r}{s}\right)^n v\right) \left(\frac{r}{s}\right)^{-n} f\left(\left(\frac{r}{s}\right)^n w\right)\right] \\ &= [T(u)T(v)T(w)], \end{aligned}$$

for all $u, v, w \in \mathcal{A}$. Thus T is a ternary homomorphism satisfying the required inequality. \square

Corollary 2.5. *Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a mapping such that $f(0) = 0$, and there exist constants $\epsilon \geq 0$ and $p < 1$ such that*

$$\begin{aligned} \left\| r f\left(\frac{\mu s x + \mu t y + [uvw]}{r}\right) - \mu s f(x) - \mu t f(y) - [f(u)f(v)f(w)] \right\| \\ \leq \epsilon(\|x\|^p + \|y\|^p + \|u\|^p + \|v\|^p + \|w\|^p), \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, and all $x, y \in \mathcal{A}$ and all $u, v, w \in \mathcal{A}$. Then there exists a unique ternary homomorphism $T : \mathcal{A} \rightarrow \mathcal{B}$ such that

$$\|f(x) - T(x)\| \leq \frac{2r^{-p}\epsilon\|x\|^p}{r^{1-p} - s^{1-p}},$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x, y, u, v, w) = \epsilon(\|x\|^p + \|y\|^p + \|u\|^p + \|v\|^p + \|w\|^p)$, and apply Theorem 2.4. \square

Remark 2.6. When $p > 1$, one may use the same techniques used in the proof of Theorem 2.4 (Theorem 2.5) to get a result similar to Corollary 2.5.

The following corollary may applied in the case that our ternary algebra is linearly generated by “idempotents”, i.e. elements u with $u^3 = u$.

Corollary 2.7. *Let \mathcal{A} be linearly spanned by a set $S \subseteq \mathcal{A}$ and $f : \mathcal{A} \rightarrow \mathcal{B}$ be a mapping with $f(0) = 0$ satisfying $f\left(\left(\frac{r}{s}\right)^{2n}[s_1 s_2 z]\right) = [f\left(\left(\frac{r}{s}\right)^n s_1\right)f\left(\left(\frac{r}{s}\right)^n s_2\right)f(z)]$ for all sufficiently large positive integers n , and all $s_1, s_2 \in S, z \in \mathcal{A}$. Suppose that there exists a function φ fulfilling (2.5), and*

$$\left\| r T\left(\frac{\mu s x + \mu t y + [uvw]}{r}\right) - \mu s T(x) - \mu t T(y) - [T(u)T(v)T(w)] \right\| \leq \varphi(x, y, u, v, w),$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in \mathcal{A}$. Then there exists a unique ternary homomorphism $T : \mathcal{A} \rightarrow \mathcal{B}$ satisfying 2.7 for all $x \in \mathcal{A}$.

Proof. By the same arguing as in the proof of Theorem 2.4, there exists a unique linear mapping $T : \mathcal{A} \rightarrow \mathcal{B}$ given by

$$T(x) := \lim_{n \rightarrow \infty} \left(\frac{r}{s}\right)^{-n} f\left(\left(\frac{r}{s}\right)^n x\right), \quad (x \in \mathcal{A}).$$

such that

$$\|f(x) - T(x)\| \leq \tilde{\varphi}(x, x, 0, 0, 0),$$

for all $x \in \mathcal{A}$. We have

$$\begin{aligned} T([s_1 s_2 z]) &= \lim_{n \rightarrow \infty} \left(\frac{r}{s}\right)^{-2n} f\left(\left(\left(\frac{r}{s}\right)^n s_1\right)\left(\left(\frac{r}{s}\right)^n s_2\right)z\right) \\ &= \lim_{n \rightarrow \infty} \left[\left(\frac{r}{s}\right)^{-n} f\left(\left(\frac{r}{s}\right)^n s_1\right)\left(\frac{r}{s}\right)^{-n} f\left(\left(\frac{r}{s}\right)^n s_2\right)f(z)\right] \\ &= [T(s_1)T(s_2)f(z)] \end{aligned}$$

By the linearity of T we have $T([xyz]) = [T(x)T(y)f(z)]$ for all $x, y, z \in \mathcal{A}$. Therefore $\left(\frac{r}{s}\right)^n T([xyz]) = T([xy\left(\left(\frac{r}{s}\right)^n z\right)]) = [T(x)T(y)f\left(\left(\frac{r}{s}\right)^n z\right)]$, and so

$$\begin{aligned} T[xyz] &= \lim_{n \rightarrow \infty} \left(\frac{r}{s}\right)^{-n} [T(x)T(y)f\left(\left(\frac{r}{s}\right)^n z\right)] \\ &= [T(x)T(y) \lim_{n \rightarrow \infty} \left(\frac{r}{s}\right)^{-n} f\left(\left(\frac{r}{s}\right)^n z\right)] \\ &= [T(x)T(y)T(z)], \end{aligned}$$

for all $x, y, z \in \mathcal{A}$. □

Theorem 2.8. Suppose that $f : \mathcal{A} \rightarrow \mathcal{B}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : A^5 \rightarrow [0, \infty)$ fulfilling (2.5), and (2.6) holds for $\mu = 1, \mathbf{i}$ and all $x \in \mathcal{A}$. Then there exists a unique ternary homomorphism $T : \mathcal{A} \rightarrow \mathcal{B}$ such that

$$\|f(x) - T(x)\| \leq \tilde{\varphi}(x, x, 0, 0, 0),$$

for all $x \in \mathcal{A}$.

Proof. Put $u = v = w = 0$ and $\mu = 1$ in (2.6). By the same arguing as Theorem 2.4 we infer that there exists a unique additive mapping $T : \mathcal{A} \rightarrow \mathcal{B}$ given by

$$T(x) := \lim_{n \rightarrow \infty} \left(\frac{r}{s}\right)^{-n} f\left(\left(\frac{r}{s}\right)^n x\right),$$

and satisfying 2.7 for all $x \in \mathcal{A}$. By the same reasoning as in the proof of the main theorem of [20], the mapping T is \mathbb{R} -linear.

Replace x by $(\frac{r}{s})^n x$, y by 0, and put $\mu = \mathbf{i}$ and $u = v = w = 0$ in (2.6). Then

$$(\frac{r}{s})^{-n} \|\frac{r}{s} f(\mathbf{i}(\frac{r}{s})^{n-1} x) - \mathbf{i} f((\frac{r}{s})^n x)\| \leq \frac{1}{s} (\frac{r}{s})^{-n} \varphi((\frac{r}{s})^n x, 0, 0, 0, 0),$$

for all $x \in \mathcal{A}$. The right hand side tends to zero as $n \rightarrow \infty$, hence

$$T(\mathbf{i}x) = \lim_{n \rightarrow \infty} (\frac{r}{s})^{-n+1} f((\frac{r}{s})^{n-1} \mathbf{i}x) = \lim_{n \rightarrow \infty} \mathbf{i} (\frac{r}{s})^{-n} f((\frac{r}{s})^n x) = \mathbf{i} T(x),$$

for all $x \in \mathcal{A}$.

For every $\lambda \in \mathbb{C}$ we can write $\lambda = \alpha_1 + \mathbf{i}\alpha_2$ in which $\alpha_1, \alpha_2 \in \mathbb{R}$. Therefore

$$\begin{aligned} T(\lambda x) &= T(\alpha_1 x + \mathbf{i}\alpha_2 x) = \alpha_1 T(x) + \alpha_2 T(\mathbf{i}x) \\ &= \alpha_1 T(x) + \mathbf{i}\alpha_2 T(x) = (\alpha_1 + \mathbf{i}\alpha_2) T(x) \\ &= \lambda T(x), \end{aligned}$$

for all $x \in \mathcal{A}$. Thus T is \mathbb{C} -linear. □

Remark 2.9. There are similar results by considering r, t instead of r, s throughout the paper (especially in Theorems 2.1 and 2.2). In the case $r = s = t = 1$, the generalized Jensen equation turns out the Cauchy equation.

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